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Endogenous discounting and the domain of the felicity function

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ABSTRACT

The objective is to show that endogenous discounting models should use a felicity function constrained to a positive domain. A variety of articles use the Mangasarian or Arrow and Kurz condition as a sufficient condition for optimality, which restricts felicity to a negative domain. Since the level of the felicity function shows up in the optimal path it leads to qualitatively different solutions when one uses a negative or positive felicity function. We suggest reasons why the domain should be positive. We furthermore derive sufficiency conditions for concavity of a transformed Hamiltonian if the felicity function is assumed to be positive.

1. Introduction

Throughout the last couple of years research on endogenous discounting models has found increasing attention in economic growth modeling. There now exist models that endogenize the discount rate via consumption (e.g. Obstfeld, 1990; Drugeon, 1996, 2001; Palivos et al., 1997; Das, 2003), environment (e.g. Pittel, 2002; Le Kama & Schubert, 2007), or human capital and others. Several articles also characterize the existence of optimal paths in the case of endogenous discounting (e.g. Becker et al., 1989; Becker and Boyd III, 1992; Sagara, 2001). Quite a number of these articles however use a felicity function with a negative domain (e.g. Palivos et al., 1997; Becker et al., 1989; Becker and Boyd III, 1992; Obstfeld, 1990; Pittel, 2002; Le Kama and Schubert, 2007).

The main reason for choosing a negative domain of the felicity function seems to be the concavity of the Hamiltonian. It can readily be shown that for all models of endogenous discounting, if one wants to prove concavity of the Hamiltonian, both the Mangasarian and the Arrow and Kurz conditions require that the domain of the felicity function is negative. We use a transformation along the lines of Nairay (1984) to show that even models with a positive felicity function can be concave under certain restrictions.

The main contribution of this article is, however, the emphasis on the sign of the felicity function. In particular, we show that the sign of the derivative of the utility functional with respect to the variable that endogenizes the discount rate crucially depends on the domain of the felicity function and argue that only a positive felicity makes sense.

The model we use is a standard Ramsey-type, infinitely-lived agent model where the discount rate is endogenized via a state variable. We choose the discount rate as a function of a state variable since in this case the importance of the domain of the felicity function is most obvious. To be precise, in this case it leads to qualitative changes in the steady state and dynamics. This type of model has been partly analyzed by Drugeon (2001), where he assumes that the discount rate is a function of consumption and capital. Though slightly more general in that respect, Drugeon's main objective is to extend the Uzawa model and study whether an endogenous discount rate can imply continuous growth. The present article, however, discusses the importance of the domain of the felicity function for several crucial features, namely steady state, dynamics, savings and speed of convergence. In several cases we extend the discussion to models where the discount rate is endogenized via consumption.

The article is organized as follows. Section 2 introduces the model and discusses necessary and sufficient conditions, steady state and dynamics. Section 3 discusses why felicity should be non-negative. Then, section 4 discusses some implications of the model and finally, the last section concludes.

2. The model

The model is based on an infinitely-lived agent approach where the agent's welfare is composed of utility from consumption and from an endogenous discount rate. In particular, we assume that

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capital affects the discount rate negatively. Since our focus here is purely on the importance of the domain of the felicity function we do not intend to provide too much empirical proof to this argument. Empirical arguments can be found in Becker and Mulligan (1997) and Frederick et al. (2002). One main reason for assuming a negative relationship between capital and the discount rate is that it is now widely known that improvements in wealth lead to lower mortality rates (see e.g. Fielding et al., 2009; Grossman, 2003; Richards and Barry, 1998). This model can therefore be understood as one where an agent maximizes utility with an uncertain lifespan.[1] We furthermore assume that capital can be accumulated by investing but is reduced by consumption and constant depreciation. The infinitely-lived agent attempts to solve the following problem.

$$\max_{\{c(t)\}} U(c(t),k(t)) = \int_{t=0}^{\infty} u(c(t))e^{-\theta(t)}dt \quad \text{subject to} \tag{1}$$

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t), \forall t \tag{2}$$

$$\dot{\theta}(t) = \rho(k(t)), \forall t \tag{3}$$

$$k(t) \geq 0, c(t) \geq 0, \forall t,$$

$$\textit{with } k(0) \textit{ given}.$$

Assumption 1. We impose that the production function $f: \mathbb{R}_+ \to \mathbb{R}_+$ follows standard assumptions of concavity, such that $f(0)=0, f'(k)>0, f''(k)<0$. We also impose the Inada conditions, given by $\lim_{k\to 0} f'(k) = \infty$, and $\lim_{k\to\infty} f'(k)=0$.

Assumption 2. The felicity function $u: \mathbb{R}_+ \to \mathbb{R}$ is at least twice continuously differentiable and has the standard properties of $u'(c)>0$, $u''(c)<0$, $\forall c$. We assume $u'(0)=\infty$.

The assumption $u'(0)=\infty$ allows to concentrate on interior solutions only. It corresponds to the assumption that at least a minimum amount of consumption is required for the representative agent. We shall also sometimes resort to the constant relative risk aversion (CRRA) felicity function, which has the functional form of $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma \geq 0$.

Assumption 3. We assume that the at least twice differentiable discount rate $\rho(k): \mathbb{R}_+ \to \mathbb{R}_{++}$, has the properties $lim_{k\to\infty}\rho(k) = \hat{\rho} > 0$, $\rho'(k)<0$, $\rho''(k)>0$, $\forall k$.

The assumption on the production and felicity function are standard. The assumptions on the discount rate help to visualize the importance of the domain of the felicity function (but can also be reconciled with empirical arguments, see Becker and Mulligan (1997) or Frederick et al. (2002)). Articles that endogenize the discount rate via a control variable are subject to the same criticisms as presented here but may not share all those features. For example, if one endogenizes the discount rate via consumption (see e.g. Obstfeld, 1990), then the steady state does not qualitatively depend on the sign of the felicity function, whereas the transition and dynamics do.

We define an admissible path as a trajectory $\{k(t), c(t), \theta(t)\}_{0\leq t\leq\infty}$ that meets the constraints (2) and (3) with states $k(t)$ and $\theta(t)$ being piecewise continuous and the control $c(t)$ piecewise continuous.

2.1. Necessary conditions

We define the Hamiltonian as follows[2]:

$$\mathcal{H} = u(c(t))e^{-\theta(t)} + \lambda(t)[f(k(t)) - c(t) - \delta k(t)] - \mu(t)\rho(k(t)). \tag{4}$$

The Pontryagin necessary conditions for optimality are

$$u'(c(t))e^{-\theta(t)} = \lambda(t), \tag{5}$$

$$\lambda(t)\left[f'(k(t)) - \delta\right] - \mu(t)\rho'(k(t)) = -\dot{\lambda}(t), \tag{6}$$

$$-u(c(t))e^{-\theta(t)} = \dot{\mu}(t), \tag{7}$$

$$\lim_{t\to\infty} H(t) = 0, \tag{8}$$

where Eq. (8) gives the transversality condition of the system.[3] As $\mu(t)$ represents the implicit value of relaxing the constraint (3) by one unit, we expect that $\mu(t)$ should give us the prospective, discounted value of changes in the discount rate. We are able to confirm this as after integrating Eq. (7) we obtain

$$\mu(t) = \int_t^{\infty} u(c(s))e^{-\theta(s)}ds. \tag{9}$$

This result is of course only valid if the utility functional converges to zero when time goes to infinity, which will be the case when felicity is finite and the discount rate is positive. Hence we know that $\mu(t)<(>)0$ if $u(c(t))<(>)0$, which is important for the second-order sufficiency conditions.

Using this indirect approach has one simple advantage: We can directly observe the shadow value of the discount rate. As we know, the multiplier represents the marginal change in the objective functional arising from an infinitesimal change in the constraint. We know that a higher discount rate reduces the value of the objective functional, suggesting that the shadow value associated to a marginal change in the discount rate should be negative. Since we introduce the term $\mu\rho(k)$ with a minus sign in the Hamiltonian, intuition suggests μ should be positive. From Eq. (9) we see that this is only reconcilable with a positive felicity function. We will elaborate on this basic result during the next pages.

2.2. Sufficiency conditions

There exists a large literature of articles that constrain the felicity function to a negative domain[4] (e.g. Epstein & Hynes, 1983; Palivos et al., 1997; Obstfeld, 1990; Pittel, 2002; Le Kama & Schubert, 2007). In several articles this assumption is used in order to insure concavity of the Hamiltonian. It can readily be shown that the Mangasarian (and equivalently Arrow and Kurz) condition for the concavity of the Hamiltonian require the domain of the felicity function to be negative (see Appendix). Since we show in this article that different assumptions on the domain of the felicity function lead to qualitative differences in models of endogenous discounting but that only felicity functions constrained to the positive domain make intuitive sense, we

[1] The equivalence derives as follows. Assume the agent maximizes $E\int_0^T u(c_t)dt$, where E is the expectations operator and T is the expected end of life. Assume that the probability of death having occurred at time t is given by the c.d.f $G(t)=P(s\leq t)=1-\theta(t)$, with $\theta(t)$ being the survival function. Then $\theta(t)=\exp\{-\int_0^t \rho_\tau d\tau\}$, and we derive $E\int_0^T u(c_t)dt = \int_0^\infty u(c_t)\exp\{-\int_0^t \rho_\tau d\tau\}dt$. This is equivalent to the control problem in 1.

[2] Defining the optimization problem by introducing the discount factor as another constraint allows the Hamiltonian to be independent of time which greatly simplifies the analysis. Throughout the article we use $x'(y)=\partial x/\partial y$, $\dot{x}=\partial x/\partial t$.

[3] For this kind of transversality condition, see Michel (1982).

[4] At the same time, there exist articles that constrain felicity to be positive, like Drugeon (1996), Epstein (1987) or Das (2003), although for different reasons.

also need to show that optimization under a positive domain leads to a maximum. We do this by reformulating the Hamiltonian along the lines of Nairay (Nairay, 1984).

We reduce the control problem to one with one state variable only. As $\frac{d\theta}{dt} = \rho(k)$, and since this is a monotonic function, we can write $dt = \frac{d\theta}{\rho(k)}$. The new Hamiltonian, denoted $\mathcal{H}_\Delta$, is now

$$\mathcal{H}_\Delta = \frac{u(c(t))e^{-\theta(t)}}{\rho(k(t))} + \lambda(t)\left[\frac{f(k(t)) - c(t) - \delta k(t)}{\rho(k(t))}\right]. \tag{10}$$

First-order condition with respect to the control variable is

$$u'(c(t))e^{-\theta(t)} = \lambda(t). \tag{11}$$

Denote this as $c(t) = \psi(\theta(t), \lambda(t))$, and substitute this back into the Hamiltonian to obtain the Hamiltonian along the optimal path. Then take second-order conditions of $\mathcal{H}_\Delta$ with respect to the state *k(t)* to see whether the Hamiltonian is concave along the optimal path. First-order conditions are

$$\frac{\partial \mathcal{H}_\Delta}{\partial k} = -\left(ue^{-\theta} + \lambda(f(k) - \psi(\theta, \lambda) - \delta k)\right)\frac{\rho'(k)}{\rho(k)^2} + \lambda\left[\frac{f'(k) - \delta}{\rho(k)}\right], \tag{12}$$

while second-order conditions are

$$\frac{\partial^2 \mathcal{H}_\Delta}{\partial k^2} = -\mathcal{H}_\Delta\left(\frac{\rho''(k)}{\rho(k)} - 2\frac{\rho'(k)^2}{\rho(k)^2}\right) + \lambda\frac{f''(k)}{\rho(k)} - 2\rho'(k)\lambda\left[\frac{f'(k) - \delta}{\rho(k)^2}\right]. \tag{13}$$

Since $\mathcal{H}_\Delta > 0$, then concavity of the Hamiltonian holds along the optimal path if $\left(\frac{\rho''(k)}{\rho(k)} - 2\frac{\rho'(k)^2}{\rho(k)^2}\right) > 0$ and $\lambda\left(\frac{f''(k)}{\rho(k)} - 2\rho'(k)\left[\frac{f'(k) - \delta}{\rho(k)^2}\right]\right) < 0$. We thus impose two conditions from now on.

Assumption 4. We assume $\rho''(k) > 2\frac{\rho'(k)^2}{\rho(k)}$.

Assumption 5. We impose $f''(k) < 2\rho'(k)\frac{f'(k) - \delta}{\rho(k)}$.

Both conditions together are sufficient for the concavity of the control problem along the optimal path. For $f'(k) < \delta$, that means to the right of the Golden Rule, Assumption 5 is always satisfied.

Assumption 4 requires a sufficiently strong convexity of the discount rate with respect to capital. This technical condition demands that for increasing *k*, the ratio between marginal changes in ρ and the deviations of ρ from zero should increase.[5] Assumption 5 requires a sufficiently strong concavity of the production function.

This sufficiency condition along the optimal path therefore replaces the condition used in the previous articles on endogenous discounting, namely the negative domain of the felicity function, with a concavity condition on the production function and a convexity condition on the endogenous discount rate. The approach suggested by Nairay (1984) and employed here is the key to obtaining a concave Hamiltonian and therefore assuring optimality.

2.3. Solving the model

Transforming the Pontryagin necessary conditions from Eq. (5) till Eq. (7) and disregarding time subscripts for convenience, we arrive at the following system of dynamic equations:

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left[f'(k) - \delta - \rho(k) - \frac{\mu}{u'(c)e^{-\theta}}\rho'(k)\right], \tag{14}$$

$$\dot{k} = f(k) - c - \delta k, \tag{15}$$

$$\dot{\mu} = -u(c)e^{-\theta}. \tag{16}$$

It is possible to reduce this system to a system in two dynamical equations, only. This we do as follows. The first observation is that the Hamiltonian of the above system is autonomous. We can therefore show the following. As we have that $\mathcal{H} = u(c)e^{-\theta} + \lambda\dot{k} - \mu\dot{\theta}$, then we know that necessary conditions for optimality are $\frac{\partial\mathcal{H}}{\partial c} = 0$, $\frac{\partial\mathcal{H}}{\partial k} = -\dot{\lambda}$, $\frac{\partial\mathcal{H}}{\partial\theta} = \dot{\mu}$, as well as $\frac{\partial\mathcal{H}}{\partial\lambda} = \dot{k}$ and finally $\frac{\partial\mathcal{H}}{\partial\mu} = \dot{\theta}$. Taking partial differentials of the Hamiltonian with respect to time we obtain $\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t} + \frac{\partial\mathcal{H}}{\partial c}\dot{c} + \frac{\partial\mathcal{H}}{\partial k}\dot{k} - \frac{\partial\mathcal{H}}{\partial\theta}\dot{\theta} + \frac{\partial\mathcal{H}}{\partial\lambda}\dot{\lambda} - \frac{\partial\mathcal{H}}{\partial\mu}\dot{\mu}$. Given the conditions for optimality we can cancel out and are left with $\frac{d\mathcal{H}^*}{dt} = \frac{\partial\mathcal{H}}{\partial t}$. As the Hamiltonian is autonomous, we also have that $\frac{\partial\mathcal{H}^*}{\partial t} = 0$. Given the transversality condition $lim_{t\to\infty}\mathcal{H}(t) = 0$, this gives us that the *optimized* $\mathcal{H}^*(t) = 0, \forall t$.

Hence we can transform the Hamiltonian to

$$\mu = \frac{u(c)e^{-\theta} + u'(c)e^{-\theta}\dot{k}}{\rho(k)}, \tag{17}$$

which we then can substitute in the system of Eqs. (14)–(15) to get the following system

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left[f'(k) - \delta - \rho(k) - \frac{\rho'(k)}{\rho(k)}\left(\frac{u(c)}{u'(c)} + \dot{k}\right)\right], \tag{18}$$

$$\dot{k} = f(k) - c - \delta k. \tag{19}$$

This reduced-form system captures the complete dynamics of the system. As is easily visible, the new term in comparison to the exogenous discounting Ramsey case is the one taking the effect of $\rho'(k)$ into account. This can be understood as an anticipatory effect: if capital is increasing in the future, thus $\dot{k} > 0$, then the agent will be inclined to increase consumption, too. Furthermore, please notice the role the sign of the felicity function plays here, an important issue which we shall return to later on.

2.4. Steady state analysis

We now analyze the steady state of this model and demonstrate the effects of assuming different signs for the felicity function. At steady state we have that $\dot{c} = \dot{k} = 0$, leading to

$$f'(k) - \delta - \rho(k) = \frac{\rho'(k)}{\rho(k)}\frac{u(c)}{u'(c)}, \tag{20}$$

$$f(k) - \delta k = c. \tag{21}$$

One can immediately spot the significance of the sign of the felicity function here. For $u(c) < 0$, the right-hand side of Eq. (20) is positive, thus constraining any steady state to be to the left of the Golden Rule[6]. However, on the contrary, for $u(c) > 0$, the right-hand side is negative, and whether a steady state is to the left of the Golden Rule or to the right depends on the sign of $\rho(k) + \frac{\rho'(k)}{\rho(k)}\frac{u(c)}{u'(c)}$. This suggests a strong result: A steady state, if it exists, is not invariant with respect to affine transformations of the felicity function. Obviously, in the Ramsey case where $\rho'(k) = 0$,

[5] A functional form that satisfies this assumption for all *k* is $\rho(k) = \exp\{\beta k^{0.05}\}$, with $\beta < 0$.

[6] The Golden Rule level is the maximum sustainable consumption level, which in this case is where $f'(k) = \delta$.

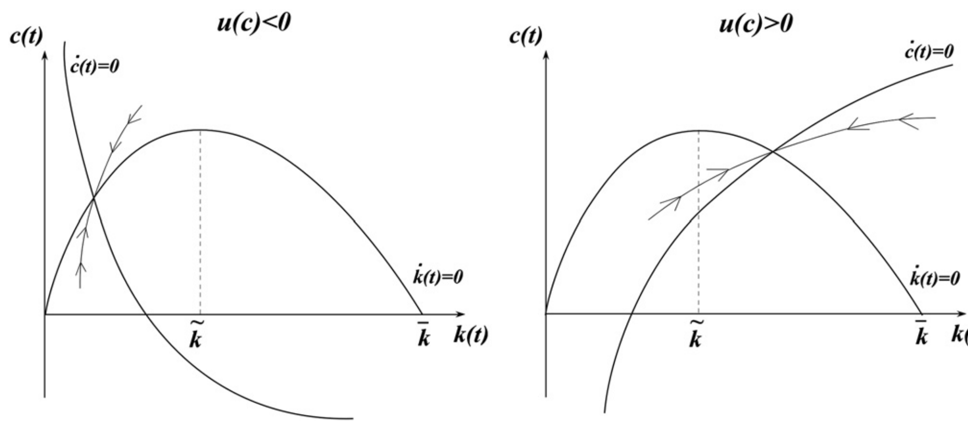

Fig. 1. Phase diagram.

the right-hand side vanishes and the sign of the felicity function plays no role in steady state selection.

Clearly, the assumption of $\rho'(k)<0$ is a necessary assumption for finding a steady state to the right of the Golden Rule. As suggested above, we only impose this assumption in order to magnify the implication of assuming different domains for the felicity function. Intuitively, however, the endogeneity of the discount rate with respect to wealth leads the agent to place more value on wealth accumulation, since this now also leads to a path with a lower discount rate. When one interprets the changes in the discount rate as reflecting changes in the mortality rate, then this implies that the agent would expect to live longer, leading to a larger valuation of felicities in the later periods. In the standard Ramsey model with exogenous discounting, a steady state to the right of the golden rule would be impossible and associated with an over-accumulation of capital. In our case, this over-accumulation is useful since capital is not only directed towards consumption, but also towards a path $\{c_t,k_t\}_{t=0}^{\infty}$ with a lower discount rate.

We will now study existence of equilibria. For simplicity we define $m(k) = \frac{\rho''(k)\rho(k)-\rho'(k)^2}{\rho(k)^2}$ with $m(k)>0$ from Assumption 4; and $n(c) = \frac{u'(c)^2-u''(c)u(c)}{u'(c)^2}$, with $n(c)>0$ if $u(c)>0$ and $n(c) \gtrless 0$ if $u(c) < 0$.

Proposition 1. Given the optimization problem defined in Eq. (1) with Assumptions 1–3, then a sufficient condition for a unique, positive steady state is

$$f''(k)<\rho'(k) + m(k)\frac{u(f(k)-\delta k)}{u'(f(k)-\delta k)} + \left(f'(k)-\delta\right)\frac{\rho'(k)}{\rho(k)}n(f(k)-\delta k), \quad (22)$$

$\forall k$.

Otherwise, multiple steady states can occur. A necessary and sufficient condition for multiple steady states is $\exists \check{k} > 0$, which solves $f'(k)-\delta-\rho(k) = \frac{\rho'(k)}{\rho(k)}\frac{u(f(k)-\delta k)}{u'(f(k)-\delta k)}$, such that

$$f''(\check{k})>\rho'(\check{k}) + m(\check{k})\frac{u\left(f(\check{k})-\delta\check{k}\right)}{u'\left(f(\check{k})-\delta\check{k}\right)} + \left(f'(\check{k})-\delta\right)\frac{\rho'(\check{k})}{\rho(\check{k})}n\left(f(\check{k})-\delta\check{k}\right). \quad (23)$$

Proof 1. We make use of the steady state Eqs. (20) and (21). Firstly, we constrain the domain of capital to the relevant one only. We do not need to concern ourselves with $k > \bar{k}$, where $\bar{k}$ is the level of capital that solves $f(k)=\delta k$, so it is sufficient to concentrate on the interval $k\in[0, \bar{k}]$ only. For simplicity later, we also define $\tilde{k}$, which solves $f'(k)=\delta$. We define $G(k)=A(k)-B(k)$, where $A(k)=f'(k)-\delta-\rho(k)$ and $B(k) = \frac{\rho'(k)}{\rho(k)}\frac{u(f(k)-\delta k)}{u'(f(k)-\delta k)}$. Then $G'(k) = f''(k)-\rho'(k)-m(k)\frac{u(c)}{u'(c)} - [f'(k)-\delta]\frac{\rho'(k)}{\rho(k)}n(k)$, where $c=f(k)-\delta k$. We know that $lim_{k\to 0}G(k)=\infty$ and $lim_{k\to\bar{k}}G(k) = z<0$, where z is a negative but finite number. Hence, the curve *G(k)* starts from positive infinity to negative finite for $k\in[0,\bar{k}]$. As each argument of *G(k)* is continuous, we know that *G(k)* is continuous. Therefore, a sufficient condition for a unique steady state is then that $G(k)=0$ only for one *k*. This is satisfied when $G'(k)<0$, $\forall k$. Multiple equilibria can only exist if *G(k)* becomes positive again after having become negative. Thus, assuming that there $\exists\check{k}$, such that $G(\check{k}) = 0$, together with $G'(\check{k}) > 0$ gives the desired condition.[7] ■

The condition for a unique steady state reduces to $f''(k)<0$ if $\rho'(k)=0$, which is the same condition as in the Ramsey case. Therefore, this model provides a tractable and direct extension of the original Ramsey case. However, as suggested above and unlike the Ramsey case, the sign of the felicity function leads to different choices of equilibria.

Remark 1. The variance of the steady states to affine transformations is also absent in the endogenizations via consumption, as in Obstfeld (1990) or Das (2003). In that case, the steady state equation reduces to $f'(k)-\delta=\rho(f(k)-\delta k)$, using our notation. However, as we shall demonstrate later, even in those models the sign of the felicity function still matters for the convergence to the steady state, and therefore leads to qualitative changes nevertheless.

2.5. The dynamics

Our intention now is to derive the phase curves as well as the conditions for stability. Here we need to carefully distinguish between positive and negative felicity. So, firstly, we remind of the steady state equations

$$f'(k)-\delta-\rho(k) = \frac{\rho'(k)}{\rho(k)}\frac{u(c)}{u'(c)}, \quad (24)$$

$$f(k)-\delta k = c. \quad (25)$$

The steady state curves have the following shape. Eq. (25) is standard and goes from $\{c,k\}=\{0,0\}$ to $\{c,k,\} = \{0,\bar{k}\}$. There exists a maximum at $f'(k)=\delta$.

For $u(c)>0$, Eq. (24) is only satisfied if, when $k\to 0$ then c should become negative; and if when $k\to\bar{k}\Rightarrow c>0$. Since *c* negative is impossible, that means there exists a $k_a>0$ such that $\forall k\leq k_a$, $c=0$. Therefore the steady state curve for consumption goes from 0 to a positive number in the interval $k\in[k_a,\bar{k}]$. Due to the continuity of

[7] For simplicity we shall avoid the case $G'(k)=0$ and $G(k)=0$.

both steady state curves for consumption and capital, this implies that a steady state always exists.

For $u(c)<0$, then Eq. (24) is only satisfied if, when $k\rightarrow 0$, then $c\rightarrow\infty$, and if $k\rightarrow\bar{k}$, then $c<0$. Therefore, there exists a maximum k_b, such that $\forall k>k_b$, $c=0$. Again, due to continuity, a steady state always exists and is unambiguously to the left of the Golden Rule.

We take the total derivative of the steady state curve of capital, Eq. (24), and consumption, Eq. (25), to get

$$\frac{dc}{dk}=\frac{f''(k)-\rho'(k)-m(k)\frac{u(c)}{u'(c)}}{n(c)\frac{\rho'(k)}{\rho(k)}}, \tag{26}$$

$$\frac{dc}{dk}=f'(k)-\delta, \tag{27}$$

where *m(k)* and *n(c)* are defined as above. Then the steady state curve (27) has the familiar shape of first increasing, reaching the maximum at $\tilde{k}$, and decreasing thereafter, until crossing the k-axis at $\bar{k}$. It is not immediately possible to provide a sign for $dc/dk|_{\dot{c}=0}$. What we do, however, know is that the phase curve for consumption must be at a higher consumption level for $k\rightarrow 0$ than for $k\rightarrow\bar{k}$ if $u(c)<0$ and vice versa if $u(c)>0$, although the slope of the phase curves in both cases need not be monotonic on the whole domain. However, if the preceding sufficiency condition for a unique steady state is satisfied then for $u(c)>0$ we have $dc/dk>0$ for $k<\tilde{k}$,and $dc/dk\gtrless 0$ for $k>\tilde{k}$, although if $dc/dk<0$, then the decrease must be slow enough to satisfy the conditions for a unique steady state.

We can show that the steady state curve for consumption reduces to the one in the Ramsey model. From Eq. (26) we obtain $dk/dc\rightarrow 0$ when $\rho'(k)\rightarrow 0$, $\forall k$, which is equivalent to the familiar vertical $\dot{c}=0$ line in the Ramsey model. Finally, the steady state curves in case of a unique steady state will approximately take the form as in Fig. 1.

Proposition 2. Given the optimization problem 1 with Assumptions 1–3, then a steady state is saddle-path stable if

$$f''(k)<\rho'(k)+m(k)\frac{u(f(k)-\delta k)}{u'(f(k)-\delta k)}+\left(f'(k)-\delta\right)\frac{\rho'(k)}{\rho(k)}n(f(k)-\delta k), \tag{28}$$

$\forall k$. It is instable otherwise.

Proof 2. We linearize the system in Eqs. (18) and (19) around the unique steady state. This gives the Jacobian of the linearized system

$$J=\begin{bmatrix}\rho(k)+\delta-f'(k) & -\frac{u'(c)}{u''(c)}\left(f''(k)-\rho'(k)-m(k)\frac{u(c)}{u'(c)}-\left(f'(k)-\delta\right)\frac{\rho'(k)}{\rho(k)}\right)\\ -1 & f'(k)-\delta\end{bmatrix}. \tag{29}$$

As is well-known, the system is saddle path stable if there exists one positive and one negative eigenvalue, denoted by $\lambda_{1,2}$. As the trace is $\mathrm{Tr}(J)=\lambda_1+\lambda_2$, and the determinant is $\mathrm{Det}(J)=\lambda_1\lambda_2$, it suffices to show that the trace is positive and the determinant is negative. We can thus show that the trace of this matrix is given by $\mathrm{Tr}(J)=\rho(k)>0$, while the determinant is negative if $f''(k)<\rho'(k)+m(k)\frac{u(c)}{u'(c)}+\frac{\rho'(k)}{\rho(k)}(f'(k)-\delta)n(k)$. ■

It thus follows trivially that if the sufficient condition for a unique steady state is satisfied then the system is saddle-path stable. Furthermore, in the case of multiple equilibria, saddle-path stable and instable equilibria will alternate, beginning with a saddle-path stable one, see Schumacher (2009).

Finally, when comparing the condition for saddle-path stability to the traditional Ramsey case, then for $\rho(k)=\rho$ the above condition reduces to $f''(k)<0$, which is again the familiar Ramsey condition.

In summary so far, although the model provides a direct extension of the Ramsey case, we notice that the steady state is not invariant with respect to affine transformations of the felicity function. The qualitative changes in the model for different signs of the felicity function however require us to choose a sign for the felicity function. In the next section we shall therefore put forward one reason why the domain of the felicity function should be constrained to be positive.

3. The sign of the felicity function

One way to decide on the sign of the felicity function is by checking under which conditions this model predicts results in the line of the Ramsey model with constant discounting. We take the steady state version of the optimal consumption Eq. (18), given by

$$f'(k)-\delta=\rho(k)+\frac{\rho'(k)}{\rho(k)}\frac{u(c)}{u'(c)}. \tag{30}$$

In the original Ramsey case, we have $\rho'(k)=0$, $\forall k$. Therefore, the smaller is ρ, the closer is the optimal level of capital to the Golden Rule level. Hence, discounting is something "bad" in a sense, which keeps one from attaining the highest possible level of consumption.[8] So now, if we take the case of $\rho'(k)<0$, then we would expect that a policy maker would try to increase the level of capital, as this brings him closer to the Golden Rule. This means that increases in capital both imply a path of $\{c_t,k_t\}_{t=0}^{\infty}$ that leads to decreases in the level of the discount rate and increases in the level of consumption over time. This is the direct feedback which we observe through the level of the discount rate itself.

In addition, we have an indirect effect, measured by the value the agent attaches to a higher utility functional from marginal decreases in the discount rate. However, the direction of this indirect effect is crucially hinging on the sign of the felicity function. If the sign is negative, then this will counterbalance the decrease in the discount rate and will make the agent choose a low capital level. Therefore, in the case of negative felicity, the agent prefers a path of $\{c_t,k_t\}_{t=0}^{\infty}$ that also leads to *increases* in the level of the discount rate. On the contrary, in the case of positive felicity, this will reinforce the direct effect and potentially even lead to a level of capital above the golden rule one. This suggests that with positive felicity, the agent prefers a path of $\{c_t,k_t\}_{t=0}^{\infty}$ that also implies decreases in the level of the discount rate.

One may think here about an applied argument. For example, re-interpret the discount rate as endogenous mortality. The higher the level of capital, the lower the mortality rate. With a negative felicity function, a higher level of capital would reduce overall welfare. Therefore, ceteris paribus, reductions in mortality through increases in capital would have a negative impact on welfare. With a positive felicity function, higher levels of capital, ceteris paribus, increase overall utility. If one wants to model the impact of capital on mortality, then one would obviously believe that a lower mortality rate should increase overall welfare. This will only be the case if the felicity function is positive, hence our emphasis on the sign of the felicity function.

It is worthwhile to note that this result is (in general) not an implication of the intertemporal elasticity of substitution like one might be inclined to believe. Clearly, if $u(c)=log(c)$ with $c\in(0,1)$, which is for example possible by normalizing capital (and subsequently having to impose further restrictions on *f(k)*), then felicity is always negative. On the converse, if $u(c)=log(c)$, with $c>1$, then felicity is always positive. Nevertheless, the intertemporal elasticity of

[8] For early references on critical approaches towards impatience c.f. Marshall (1890), Pigou (1920) or Ramsey (1928).

substitution is one in both cases. We therefore must strictly exclude the argument that this result is driven by the level of the intertemporal elasticity of substitution.

To provide more foundation to this we transform Eq. (14) by substituting for $\mu(t)$ its optimal value, which is the discounted prospective value of utility on the optimal path. We therefore obtain

$$\dot{c}(t) = -\frac{u'(c)}{u''(c)}\left[f'(k(t)) - \delta - \rho(k(t)) - \frac{\rho'(k(t))\int_t^\infty u(c(s))e^{-\theta(s)}ds}{u'(c(t))e^{-\theta(t)}}\right]. \quad (31)$$

We are going to analyze the last part of this equation, which helps us in understanding the paradox raised in the previous paragraphs:

$$\frac{-\rho'(k(s))e^{-\int_0^t \rho(k_s)ds}\int_t^\infty u(c(s))e^{-\int_t^s \rho(k(\tau))d\tau}ds}{u'(c(t))e^{-\int_0^t \rho(k_s)ds}}. \quad (32)$$

The term above is the ratio of two Volterra derivatives. The denominator is the Volterra derivate with respect to consumption, which is always positive, implying that higher consumption has a positive marginal effect on the utility functional. The numerator of this ratio is the Volterra derivative of the utility functional with respect to capital. Given negative felicity the numerator is negative, suggesting that higher capital reduces overall utility. Thus, in the case of endogenous discounting with negative felicity, discounting becomes something "good" in the sense that higher discounting actually *increases* the utility functional. Mathematically, we obtain that for $V(k) = \int_0^\infty u(c)exp\{\int_0^t \rho(k)ds\}dt$ the result that $\partial V(k)/\partial k > 0$ if $u(c) > 0$, implying that – ceteris paribus – increases in capital increase the total welfare. On the converse, if $u(c) < 0$ then $\partial V(k)/\partial k < 0$, suggesting that capital increases would reduce overall welfare. This, of course, is the result of having the level of felicity itself in the Volterra derivative. We therefore do not only have to look at marginal changes, but also at the sign of the felicity function. Conclusively, we lose invariance with respect to affine transformations. Specifically, felicity functions give rise to opposite conclusions depending on whether felicity is negative or positive, independently of the sign of the discount rate and the effect of capital on consumption.

Remark 2. Imagine that instead of endogenous discounting we had capital in the felicity function, such that $u(c, k)$, with $u_k > 0$. The steady state equation for consumption would become $f'(k) - \delta = \rho + u_k/u_c$. Then, the last term u_k/u_c would intuitively play the same role as the ratio of the Volterra derivatives above. Increases in capital increase overall utility leading to a higher steady state level of capital. The argument here is clearly the same as in the endogenous discounting case above, with the difference that the sign of the felicity function plays no role per se.

Remark 3. Taking the case of a discount rate increasing in consumption then the Volterra derivative with respect to consumption, evaluated from time t onwards, will be given by $u'(c_t) - \rho'(c_t)\int_t^\infty u(c_s)e^{-\int_t^s \rho(c_\tau)d\tau}ds$. Again, the sign of the felicity function determines the effect on total welfare. If $\rho'(c) > 0$ as in Obstfeld (1990) then increases in consumption bear two effects: the effect on utility today which is increasing my overall welfare, versus the effect of an increasing discount rate, which intuitively ought to diminish my overall welfare. In the case of a negative felicity then more consumption increases the utility functional, even though it increases the rate of impatience. It seems difficult to reconcile this result with intuition. In addition, the standard constraint that $u'(c) - \rho'(c_t)\int_t^\infty u(c_s)e^{-\int_t^s \rho(c_\tau)d\tau}ds > 0$ seems too strong. Since this term represents the trade-off of current consumption versus impacts on future valuations, one should certainly allow future valuations to be more important for the policy maker than current consumption.

Remark 4. The analysis above is not constrained to the assumption that $\rho'(k) < 0$. Indeed, let us, for the purpose of illustration only, now assume that $\rho'(k) > 0$. In this case, if the representative agent chooses a path with a higher capital stock, then this path will also have a higher discount rate. Intuitively, an increasing capital stock will benefit the agent through a potentially higher consumption in future periods, while it should hurt him through a higher discount rate. This reasoning is, however, only confirmed for a positive felicity. For example, in the first-order condition (6), the term $\rho'(k)\mu$ is positive for $u > 0$, and therefore, ceteris paribus, an increase in capital reduces this term (for a concave $\rho(k)$) which has a negative effect on the change in the value of relaxing the capital constraint by one unit (thus $\dot{\lambda}$). While this is certainly a reasonable result, a negative felicity function would lead to the opposite conclusion.

4. Discussion and analysis

This section will utilize the results of the model to assess several specific questions. Firstly, what is the effect of changing technology? Secondly, what can be said about the savings behavior in this recursive model? Thirdly, what implications has this model on the convergence speed? We shall analyze the cases with emphasis on positive felicity.

4.1. The effect of technology

From the steady state equations we can also draw some conclusions on the effect of a higher level of total factor productivity, A, where we write $f(k) = Ag(k)$, with $A > 0$. Mathematically, we take partial derivatives of Eqs. (24) and (25) with respect to k and A to get

$$\frac{dc}{dA} = \frac{\rho(k)g'(k)u'(c)^2}{\rho'(k)\left[u'(c)^2 - u''(c)u(c)\right]},$$
$$\frac{dc}{dA} = g(k) > 0.$$

Thus, in the case of $u(c) > 0$, for increases in total factor productivity the $\dot{c} = 0$ shifts down whereas the $\dot{k} = 0$ line expands. The intuition is that if capital is more productive then this firstly allows a higher exogenous level of capital and secondly, as it increases the welfare functional through a smaller discount rate, it provides incentive to accumulate more capital. In comparison, in the case of $u(c) < 0$, it is not clear in which direction the phase curve for consumption shifts.

4.2. Effect on savings behavior

The saving rate is given by $s = 1 - c/f(k)$. We can then calculate the steady state savings rate to assess what influence the endogenous discounting has on savings. We shall do this for the familiar Cobb-Douglas production function, in which case $f(k)/k = f'(k)/\alpha$ holds, as well as for the CRRA felicity function. Let us denote steady state savings by s^*, then it is possible to calculate the steady state savings rate as

$$s^*_{en} = \frac{\alpha\delta}{\delta + \rho(k) + \frac{\rho'(k)}{\rho(k)}\frac{f(k) - \delta k}{1 - \sigma}}. \quad (33)$$

We compare this to the steady state savings rate in the exogenous discounting case, which is $s^*_{ex} = \frac{\alpha\delta}{\delta + \rho}$. Whether steady state savings in the endogenous discounting case is higher than in the exogenous one now fully depends on whether $\rho(k)$ is larger or smaller than in the exogenous case, as well as how the representative agent values marginal changes in the discount rate. Assuming, for simplicity, that both discount rates are equal, then the steady state saving rate in the endogenous discounting case is higher than in the exogenous discounting case only if $\sigma < 1$, and lower if $\sigma > 1$. Therefore, the sign of the felicity function again plays a clear role. Negative CRRA felicity leads to unambiguously lower savings in the steady state than positive felicity.

We now analyze the steady state savings rate for changes in the intertemporal elasticity of substitution (IES), but constraining the IES to the relevant range only, $\sigma\in(0, 1)$. We know that Eq. (30) must be satisfied at steady state. We then calculate the change in steady state capital when there is a change in the IES. We do this by taking total derivatives to obtain

$$\frac{dk}{d\sigma} = \frac{\frac{\rho'(k)}{\rho(k)}\frac{f(k)-\delta k}{(1-\sigma)^2}}{f''(k)-\rho'(k)-m(k)\frac{f(k)-\delta k}{1-\sigma}-\frac{\rho'(k)}{\rho(k)}\frac{f'(k)-\delta}{1-\sigma}}. \tag{34}$$

The nominator is always negative and the denominator is negative in the vicinity of a saddle-path stable steady state.[9] If the sufficient condition for a unique steady state is satisfied, then $dk/d\sigma>0, \forall k$.

We are now in a position to retrieve the total effect of the IES on the steady state savings rate. We take the total derivate of s^*_{en} and obtain

$$\frac{ds^*_{en}}{d\sigma} = -\frac{\alpha\delta}{\left(\delta+\rho(k)+\frac{\rho'(k)}{\rho(k)}\frac{f(k)-\delta k}{1-\sigma}\right)^2} \left[\left(\rho'(k)+m(k)\frac{f(k)-\delta k}{1-\sigma}+\frac{\rho'(k)}{\rho(k)}\frac{f'(k)-\delta}{1-\sigma}\right)\frac{dk}{d\sigma}+\frac{\rho'(k)}{\rho(k)}\frac{f(k)-\delta k}{(1-\sigma)^2}\right], \tag{35}$$

where k is evaluated at steady state. A sufficient condition for $ds^*_{en}/d\sigma>0$ then is $m(k)\leq 0$ and $f'(k)>\delta$.

4.3. Changes in convergence speed

As is well known, the convergence speed in the original Ramsey model is much too high to match the data. Hence, we would like to know whether this model is able to generate a convergence speed more in line with the actual data.

We can approximate the convergence speed with the negative eigenvalue of the linearized system. We know that the trace and determinant are given by $\mathrm{Tr}(J)=\rho(k)>0$ and $\mathrm{Det}(J)=f''(k)-\rho'(k)-m(k)\frac{u(c)}{u'(c)}-\frac{\rho'(k)}{\rho(k)}(f'(k)-\delta)n(k)$. We then denote the negative eigenvalue by $\lambda^{en}<0$, which is given by the formula

$$\lambda^{en} = \frac{\rho(k)}{2}-\sqrt{\frac{\rho(k)^2}{4}+\frac{u'(c)}{u''(c)}\left(f''(k)-\rho'(k)-m(k)\frac{u(c)}{u'(c)}-\frac{\rho'(k)}{\rho(k)}(f'(k)-\delta)n(k)\right)}. \tag{36}$$

In comparison, the convergence speed in the Ramsey case, which we denote by λ^{ex} for exogenous discounting, is approximately

$$\lambda^{ex} = \frac{\rho}{2}-\sqrt{\frac{\rho^2}{4}+\frac{u'(c)f''(k)}{u''(c)}}. \tag{37}$$

We shall use specific functional forms as well as common parameter ranges in order to draw some conclusions. The functional forms are $f(k)=Ak^{\alpha}$, $\rho(k)=\frac{\rho}{1+\beta k}$, $u(c)=\frac{c^{1-\sigma}}{1-\sigma}$, with parameters given by $A=2, \alpha=0.3, \beta=0.03, \delta=0.05, \rho=0.3, \sigma_1=0.9$ for positive felicity and $\sigma_2=1.1$ for negative felicity.

Mankiw (1995), Phelps and Romer estimate the convergence speed for the United States. Their rough estimate is at around 4%. In comparison to this the convergence speed with the exogenous discounting Ramsey model is approximately 42%, which is much too high. The endogenous discounting model presented here gives an approximate convergence speed of 3% for positive felicity, which is much more in line with the empirical data. The same model with negative felicity however gives a convergence speed of approx. 12%. Therefore, small changes in the intertemporal elasticity of substitution which imply a change in the sign of the felicity function may lead to significant changes in the convergence speed.

Remark 5. Even though the sign of the felicity function does not affect the steady state when the discount rate is endogenized via consumption, we nevertheless see an effect on the convergence speed. It is straight-forward to calculate the convergence speed in that model, which is equal to

$$\kappa = \frac{\rho(c)}{2}-\sqrt{\frac{\rho^2}{4}-(f''(k)-\rho'(c)\rho(c))\left(\frac{u'(c)\rho(c)-\rho'(c)u(c)}{-u''(c)\rho(c)+\rho''(c)u(c)}\right)}.$$

As discussed in Das, 2003, saddle-path stability in her model where $u(c)>0$ requires $f''(k)<\rho'(c)\rho(c)$. On the contrast, negative felicity would introduce further conditions for saddle-path stability and different results for convergence. Interestingly, it is exactly the negative felicity which is so generally used in these models which can, for intermediate values, create the instability.

5. Conclusion

In the previous sections we argued that the felicity function should be constrained to a positive domain when one uses endogenous discounting models. Future research should base the choice of the sign of the felicity function more stringently on the part of the Volterra derivative that arises solely through the endogenized discount rate. At the same time one has to attempt to understand the meaning of felicity functions in endogenous discounting models. Since the felicity functions are not invariant with respect to affine transformations any longer, it is for example important to understand how to arrive at the global utility functional from microeconomic, revealed preferences.

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Appendix

To assess the concavity of the Hamiltonian we make use of the Mangasarian conditions. However, even the weaker Arrow and Kurz conditions have the same requirement on felicity. The Mangasarian sufficiency conditions employed here are as follows. If an admissible pair $(c^*(t), k^*(t))$ satisfies the conditions $\mathcal{H}(t, k^*(t), c(t), \lambda(t), \mu(t), \theta(t))\leq\mathcal{H}(t, k^*(t), c^*(t), \lambda(t), \mu(t), \theta(t))$, $\forall c(t)$ admissible, if the first-order conditions and the transversality conditions are satisfied, and if $\mathcal{H}(t, k(t), c(t), \lambda(t), \mu(t), \theta(t))$ is concave in $(c(t), k(t))$, then the pair $(c^*(t), k^*(t))$ is optimal.

We state the result in the next proposition.

[9] The denominator is the slope of the function $G(k)$, as defined in Proof 1.

Proposition 3. Given the optimization problem in Eq. (1) a necessary condition for the Mangasarian sufficient conditions to be satisfied is given by $u(c(t)) \leq 0$.

Proof 1. The second order sufficiency conditions are derived from the following matrix

$$M = \begin{bmatrix} u^{''}(c(t))e^{-\theta(t)} & 0 & -u^{'}(c(t))e^{-\theta(t)} \\ 0 & \lambda(t)f^{''}(k(t)) - \mu(t)\rho^{''}(k(t)) & 0 \\ -u^{'}(c(t))e^{-\theta(t)} & 0 & u(c(t))e^{-\theta(t)} \end{bmatrix}.$$

The requirement for negative definiteness is that the principal minors alternate in sign. Thus, negative (semi-) definiteness of M requires $u(c) \leq 0$. ■

Corollary 1. The above result can be directly applied to any model with endogenous and exponential discounting when one utilizes the concavity of the Hamiltonian. Assume the welfare functional is $\int_0^\infty u(x)e^{-\theta}dt$, where x is any variable and $\dot{\theta} = \rho(x)$, then the second derivative with respect to θ is always $u(x)exp\{-\theta\}$. A necessary condition for the negative definiteness of a Hessian requires the terms on the diagonal to be negative. This thus demands $u(x) < 0$.

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